Lecture 9

Example: A Lotka-Volterra Competition Model with Diffusion

Consider

$$(9.1)\quad \begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1 + u_1[a_1 - u_1 - b_1 u_2] & \text{in } \Omega \times (0,\infty) \\ \dfrac{\partial u_2}{\partial t} = \Delta u_2 + u_2[a_2 - b_2 u_1 - u_2] & \\ u_1 = 0 = u_2 & \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

As in Lecture 5, we may employ the theory of analytic semi-groups, fractional power spaces and parabolic partial differential equations to convert (9.1) into a semi-dynamical system

$$\pi((u_1^0, u_2^0), t),$$

where $\pi((u_1^0, u_2^0), t)$ denotes the unique solution of (9.1) so that

$$(u_1(x,0), u_2(x,0)) = (u_1^0(x), u_2^0(x))$$

on $\bar{\Omega}$. Here a suitable choice of function space is

$$C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$$

the Cartesian product of smooth real-valued functions on $\bar{\Omega}$ which vanish on $\partial\Omega$, equipped with the norm

$$\|(f,g)\|_{C'(\bar{\Omega})\times C'(\bar{\Omega})} = \|f\|_{C'(\bar{\Omega})} + \|g\|_{C'(\bar{\Omega})}$$

where $$\|f\|_{C'(\Omega)} = \sup_{x\in\bar{\Omega}} |f(x)| + \sum_{i=1}^{\dim\Omega} \left[ \sup_{x\in\bar{\Omega}} |\partial_{x_i} f| \right]$$

Since the components of $\pi((u_1^0, u_2^0), t)$ represent population densities on $\bar{\Omega}$, we restrict our ourselves to the cone $K$ of $C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$ where both components are nonnegative on $\bar{\Omega}$. $K$ is a complete metric space, as is required in the set-up to get a global attractor.

We next want to establish that $\pi((u_1^0, u_2^0), t)$ is dissipative. To this end, we begin with the diffusive logistic model

(9.2 $$\frac{\partial u_i}{\partial t} = \Delta u_i + u_i(a_i - u_i) \qquad \text{in } \Omega \times (0,\infty)$$

$$u_i = 0 \qquad \text{on } \partial\Omega \times (0,\infty)$$

which describes the population dynamics of species $i$

in the absence of species $j$.

Theorem 9.1. (CC Thm 4.4) Consider the eigenvalue problem

$$(9.3)\qquad \begin{aligned} \Delta w_i + a_i w_i &= \sigma_i w_i && \text{in } \Omega \\ w_i &= 0 && \text{on } \partial\Omega \\ w_i(x) &> 0 && \text{in } \Omega \end{aligned}$$

corresponding to the linearization of (9.2) about $u_i = 0$.

(i) If $\sigma_i > 0$ in (9.3), there is a unique $\tilde{w}_i \in C_0^1(\bar{\Omega})$ with $\tilde{w}_i(x) > 0$ for $x \in \Omega$ and $\frac{\partial \tilde{w}_i}{\partial \eta}(x) < 0$ on $\partial\Omega$ such that for any $w_i^0 \in C_0^1(\bar{\Omega})$ with $w_i^0 \geq, \not\equiv 0$ in $\bar{\Omega}$ the unique solution $w_i(x,t)$ of (9.2) with $w_i(x,0) = w_i^0(x)$ converges in the norm of $C_0^1(\bar{\Omega})$ to $\tilde{w}_i(x)$ as $t \to \infty$. This $\tilde{w}_i(x)$ is an equilibrium solution to (9.2) and satisfies the pointwise bound $\tilde{w}_i(x) \leq a_i$

(ii) If $\sigma_i \leq 0$ in (9.3), 0 is the only nonnegative equilibrium solution of (9.2) and for any $w_i^0 \in C_0^1(\Omega)$ with $w_i^0 \geq 0$ the unique solution $w_i(x,t)$ of (9.2) with $w_i(x,0) = w_i^0(x)$ converges in the norm of $C_0^1(\bar{\Omega})$ to $0$ as $t \to \infty$.

(iii) $\sigma_i > 0$ in (9.3) $\Leftrightarrow$ $a_i > \lambda_0^1(\Omega)$ where $\lambda_0^1(\Omega)$ denotes the principal eigenvalue of the negative Laplacian on $\Omega$ subject to zero Dirichlet boundary conditions.

Notes: (i) Propositions 8.1 and 8.2 give the result in the topology of $C(\bar{\Omega})$ as in the discussion of (8.7). We can readily extend the results to $C^1(\bar{\Omega})$ by means of Lemma 5.5 in the notes and the observation that $\frac{\partial w_i}{\partial \eta}(x,t) < 0$ on $\partial\Omega$, which follows from the Hopf or strong maximum principle.

(ii) As before $\sigma_i$ can be regarded as the average intrinsic growth rate of species $i$ over $\Omega$.

(iii) That $\hat{w}_i(x) \le a_i$ follows as in Lecture 6.

(iv) If $\varepsilon_i > 0$ and $w_i^0(x) \ge 0$ in $C_0^1(\bar{\Omega})$ are prescribed, and $w_i(x,t)$ is the unique solution of (9.2) with $w_i(x,0) = w_i^0(x)$, there is a $t_i = t_i(\varepsilon_i, w_i^0(x))$ so that $w_i(x,t) \le a_i + \varepsilon_i$ for all $t \ge t_i$.

Suppose now that $(u_1(x,t), u_2(x,t))$ is the unique solution to (9.1) with $(u_1(x,0), u_2(x,0)) = (u_1^0(x), u_2^0(x))$, where $u_i^0(x) \ge 0$ in $C_0^1(\bar{\Omega})$. Then

$$\frac{\partial u_i}{\partial t} \le \Delta u_i + u_i(a_i - u_i) \quad \text{in } \Omega \times (0,\infty)$$

The method of upper and lower solutions $\Rightarrow$ $u_i(x,t) \le w_i(x,t)$, where $w_i(x,t)$ is the unique solution to (9.2) with $w_i(x,0) = u_i^0(x)$. So there is a $\bar{t}$ depending only on $(u_1^0, u_2^0)$ so that

(9.4) $$u_i(x,t) \le a_i + 1$$

for all $t \ge \bar{t}$, for $i = 1, 2$.

Now (9.4) $\Rightarrow$ the solution corresponding to $(u_1^0, u_2^0) \in K$ is eventually bounded in the topology of $C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$. Dissipativity of $\Pi$, however, requires the bound be in the topology of $C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$. This follows from Theorems 5.1 and 5.2 and our discussion, as does the pre-compactness of $\Pi((\cdot,\cdot),t) : K \to K$ for any $t > 0$.

We may now invoke Bilotti and LaSalle (1971) to assert the existence of a global attractor $A$ for $\Pi$, contained in $K$ the positive cone of $C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$. So now if $\varepsilon > 0$ is prescribed, then for any bounded set $V$ in $K$, there $\exists$ a $t_V > 0 \ni \Pi(V,t) \subseteq \mathcal{B}(A,\varepsilon)$ for all $t \geq t_V$. Consequently, to determine the long term behavior of $\Pi$, it suffices to consider its restriction to $\mathcal{B}(A,\varepsilon)$. To this end, define $\tilde{X}$

$$\tilde{X} = \overline{\Pi(\mathcal{B}(A,\varepsilon), [t_0,\infty))}$$

and define $X$ by

$$X = \pi(\hat{X}, t')$$

for some $t' > 0$. Then $\tilde{X}$, $X$ are compact and forward invariant under $\pi$, and so are $S$ and $X \setminus S$, where $S = X \cap \partial K$. Our choice of $X$ guarantees that if $(u_1, u_2) \in S$, either $u_1 \equiv 0$ on $\Omega$ or $u_2 \equiv 0$ on $\Omega$.

Theorem 9.1 enables us to conclude that $\omega(S)$ $= \{\omega((u_1^0, u_2^0)) \mid (u_1^0, u_2^0) \in S\} = \{(0,0), (\hat{w}_1(x), 0), (0, \hat{w}_2(x))\}$ so long as $\sigma_1$ and $\sigma_2$ in (9.3) are both positive. If $\sigma_1 \leq 0$ or $\sigma_2 \leq 0$ a prediction of coexistence in (9.1) is not possible. (In this case, $\Omega$ lacks sufficient core area relative to the dissipative effect of the absorbing boundary condition.) Theorem 9.1 also guarantees that we can not produce orbits to our system that link the elements of $\omega(S)$ together in a cyclic fashion. Moreover, the elements of $\omega(S)$ are isolated invariant sets with respect to the restriction of $\pi$ to $S$.

We will say that (9.1) is <u>permanent</u> (if you like, <u>ecologically permanent</u>) if there are positive numbers $m$ and $M$ with $m < M$ so that if $(u_1(x,t), u_2(x,t))$ denotes the solution trajectory of (9.1) corresponding to initial condition $(u_1^0, u_2^0) \in C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$ with $u_i^0 \geqq \not\equiv 0$ for $i = 1, 2,$ there is a $t_0 = t_0((u_1^0, u_2^0)) > 0$ so that for $t \geq t_0$

$$m e(x) \leq u_i(x,t) \leq M$$

for all $x \in \bar{\Omega}$. Here $e(x) > 0$ is the unique solution of

$$(9.5) \qquad \begin{aligned} -\Delta u &= 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

(Because of the Dirichlet boundary condition, we can not use a constant $m$ as our "ecological floor". A positive constant $M$ will suffice as our "ecological ceiling").

Theorem 4.3 of CC (See also Cantrell et al (1993a) will guarantee that (9.1) is permanent provided that

$$\begin{aligned} &W^s(\{(0,0)\}) \cap \text{int}\, K = \phi \\ (9.6) \quad &W^s(\{(\hat{w}_1(x), 0)\}) \cap \text{int}\, K = \phi \\ &W^s(\{(0, \hat{w}_2(x))\}) \cap \text{int}\, K = 0 \end{aligned}$$

where $\text{int}\, K = \{(f_1, f_2) \in K \mid f_i(x) > 0$ for

$x \in \Omega,\ \frac{\partial f_i}{\partial \eta}(x) < 0$ on $\partial\Omega,\ i = 1, 2\}$

References on permanence:
(i) Hofbauer and Sigmund (1989, 1998)
(ii) Hutson and Schmitt (1992)
(iii) Cantrell and Cosner (2003), Chapters 4/5.

To establish (9.6) we will employ the following Lemma.

<u>Lemma 9.2.</u> (CC Lemma 4.5) Suppose $f \in C^\alpha(\Omega)$ and $\mu > 0$. Let $\sigma$ be the unique eigenvalue of

$$(9.7) \quad \begin{aligned} \mu \Delta\phi + f(x)\phi &= \sigma\phi \quad \text{in } \Omega \\ \phi &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

possessing an eigenfunction $\phi > 0$. Assume $\sigma > 0$.

Suppose for some $\varepsilon \in (0,\sigma)$, $t_* \geq 0$ and $\delta > 0$

the function $u(x,t)$ satisfies

$$(9.8) \qquad \frac{\partial u}{\partial t} \geq \mu \Delta u + [f(x) - \varepsilon] u$$

for $t \in (t_*, t_* + \delta)$. If there is a $k > 0$ so that

$$u(x,t_*) \geq k \phi(x)$$

for $x \in \bar{\Omega}$, then $u(x,t) \geq k e^{(\sigma-\varepsilon)(t-t_*)} \phi(x)$ for

$t \in (t_*, t_* + \delta)$.

(The existence of such a $k > 0$ is guaranteed by the strong (Hopf) maximum principle if $u(x,0) \geq 0$, $u(x,0) \not\equiv 0$ and $t^* > 0$.)

Proof: Set $v(x,t) = k e^{(\sigma-\varepsilon)(t-t_*)} \phi(x)$.

Then for $t > t_*$,

$$\frac{\partial v}{\partial t} - \mu \Delta v - [f(x) - \varepsilon] v$$

$$= (\sigma - \varepsilon) k e^{(\sigma-\varepsilon)(t-t_*)} \phi(x)$$

$$- \mu k e^{(\sigma-\varepsilon)(t-t_*)} \Delta \phi - [f(x) - \varepsilon] k e^{(\sigma-\varepsilon)(t-t^*)} \phi(x)$$

$$= k e^{(\sigma - \varepsilon)(t - t^*)} [\sigma \phi - f(x)\phi - \mu \Delta \phi] = 0$$

Since $u(x, t_*) \geq k\phi(x)$ for $x \in \bar{\Omega}$, the result follows from the method of upper and lower solutions.

Now consider $\{(0,0)\}$, assuming $\sigma_1$ and $\sigma_2$ in (9.3) are both positive. Let $w_i$ be as in (9.3). Then

$$\Delta w_i + (a_i - \varepsilon_i) w_i = (\sigma_i - \varepsilon_i) w_i \qquad \text{in } \Omega$$

for $\varepsilon \in (0, \sigma_i)$. Suppose now that $(u_1(x,t), u_2(x,t))$ is a solution to (9.1) (or positive orbit for $\pi$) with

$$u_i(x, t_*) + b_i u_j(x, t_*) < \varepsilon_i / 2$$

for some $t_* > 0$, $i = 1, 2$, $j \neq i$. Continuity of $\pi \Rightarrow$ $\exists\ \delta_i > 0 \ni$

$$u_i(x,t) + b_i u_j(x,t) < \varepsilon_i$$

for $t \in (t_*, t_* + \delta_i)$, $i = 1, 2$, $j \neq i$. For such $t$, $u_i$ satisfies (9.8) with $\mu = 1$, $f(x) = a_i$

and $\varepsilon = \varepsilon_i$. Since $u_i(x, t_*) > 0$ for $x \in \Omega$ and

$\frac{\partial u_i}{\partial \eta}(x, t_*) < 0$ for $x \in \partial\Omega$, $\exists\, k_i > 0 \ni$

$$u_i(x, t_*) \geq k_i w_i(x)$$

for $x \in \bar{\Omega}$. Lemma 9.2 $\Rightarrow$

$$u_i(x,t) \geq k_i e^{(\sigma_i - \varepsilon_i)(t - t_*)} w_i(x)$$

for $t \in (t_*, t_* + \delta_i)$. So each species density increases exponentially or faster whenever both densities are small, so species $i$ can invade when $u_j \equiv 0$.

So $\{(0,0)\}$ is an isolated invariant set for $\pi$ on $K$ with $W^s(\{(0,0)\}) \cap \text{int } K = \phi$.

So now consider $\{(\tilde{w}_1(x), 0)\}$) The second equation in (9.1) can be expressed as

$$\frac{\partial u_2}{\partial t} = \Delta u_2 + u_2(a_2 - b_2 \tilde{w}_1 - b_2(u_1 - \tilde{w}_1) - u_2)$$

so that (9.8) holds with $\mu = 1$, $f(x) = a_2 - b_2 \tilde{w}_1(x)$

so long as $b_2(u_1(x,t) - \tilde{w}_1(x)) + u_2(x,t) < \varepsilon$.

Let $\sigma_3$ denote the average growth rate over $\Omega$ for species 2 when $u_1 \equiv \hat{\tilde{w}}_1$; i.e., $\sigma_3$ is the principal eigenvalue in

$$\Delta w_3 + w_3 (a_2 - b_2 \hat{w}_1(x)) = \sigma_3 w_3 \quad \text{in } \Omega$$

(9.9)
$$w_3 = 0 \quad \text{on } \partial\Omega$$

$$w_3 > 0 \quad \text{in } \Omega$$

If $\sigma_3 > 0$ in (9.9) and $\varepsilon \in (0, \sigma_3)$, Lemma 9.2

$\Rightarrow$ if $b_2(u_1(x, t_*) - \hat{w}(x)) + u_2(x, t_*) < \varepsilon$

and $u_2(x, t_*) \geq K w_3(x)$ on $\bar{\Omega}$ for some $t_* \geq 0$,

then $u_2(x,t) \geq K e^{(\sigma_3 - \varepsilon)(t - t_*)} w_3(x)$ for as long after $t_*$

as $b_2(u_1(x,t) - \hat{w}_1(x) + u_2(x,t)) < \varepsilon$ in $\bar{\Omega}$ continues to hold. So the density of species 2 increases at least exponentially whenever it is small and the density of species 1 is near $\hat{w}_1(x)$, its equilibrium in the absence of competition.

So species 2 can invade species 1 at equilibrium and $\{(\hat{w}_1(x), 0)\}$ is an isolated invariant set for $\pi$ on $K$

with $W^s(\{(\tilde{w}_1(x,0)\}) \cap \operatorname{int} K = \phi$.

Species 1 can invade species 2 at equilibrium, $\{(0, \hat{w}_2(x))\}$ is an isolated invariant set for $\pi$ on $K$ and $W^s(\{(0, \hat{w}_2(x))\}) \cap \operatorname{int} K = \phi$ provided that the average growth rate $\sigma_4$ for species 1 over $\Omega$ when $u_2 \equiv \hat{w}_2(x)$ is positive. Here $\sigma_4$ is the principal eigenvalue of

$$
(9.10) \qquad \begin{aligned}
\Delta w_4 + w_4(a - b_1 \hat{w}_2(x)) &= \sigma_4 w_4 && \text{in } \Omega \\
w_4 &= 0 && \text{on } \partial\Omega \\
w_4 &> 0 && \text{in } \Omega
\end{aligned}
$$

Assuming $\sigma_1, \sigma_2, \sigma_3$ and $\sigma_4$ are all positive, the Acyclicity Theorem of Persistence Theory (Hale and Waltman 1989) $\Rightarrow$ (9.1) is permanent.

<u>Connection to Eigenvalues</u>

The condition for permanence in (9.1) is that in a suitable number of cases (here 4), the average growth rate of one of the species was positive when the remaining species' densities had a prescribed configuration. The positivity of the average rate of growth of the distinguished species has the interpretation that the prescribed configuration of the remaining species' densities is invasible by the distinguished species.

The average growth rates correspond to principal eigenvalues for certain linear operators which depend on species' densities, interaction parameters and environmental parameters. The positivity of all the eigenvalues in question is the condition we require for a finding of permanence. It is a sharp condition in the sense that if one of the eigenvalues in question is negative, it is not only the case that we are unable to predict persistence. We actually know the system is not permanent. The reason is that a result corresponding to Lemma 9.2 in the case of a negative principal eigenvalue can be used to show that a small species density lies below a positive function which decays exponentially. Hence the species in question is driven to extinction if its density becomes too low, ruling out a prediction of coexistence for the system. We should point out that this does not mean that the model predicts extinction of at least one species in the community whatever the initial configuration of the species' densities.

The second feature of this way of conceptualizing a condition for permanence results from the connection between the eigenvalues measuring average growth (the $\sigma$'s) and the companion principal eigenvalues for weighted eigenvalue problems (the $\lambda_+^1(m)$'s). As an example, $\sigma_3$ in (9.9) and $\sigma_4$ in (9.10) are positive $\Longleftrightarrow$

(9.11) $$\lambda_+^1(a_2 - b_2\tilde{w}_1(x)) < 1$$

(9.12) $$\lambda_+^1(a_1 - b_1\hat{w}_2(x)) < 1,$$

respectively. Such a connection makes it possible to examine explicitly how predictions of coexistence in the model relate to or depend on the parameters in the model. Such examinations are at the heart of the ecological analyses in CC. To this end, let us consider (9.1).

We can re-write (9.9) and (9.10) as

$$(9.13) \qquad -\Delta w_3 + b_2 \hat{w}_1(x) w_3 = (a_2 - \sigma_3) w_3 \qquad \text{in } \Omega$$

$$(9.14) \qquad -\Delta w_4 + b_1 \hat{w}_2(x) w_4 = (a_1 - \sigma_4) w_4 \qquad \text{in } \Omega$$

where $w_3, w_4$ are positive in $\Omega$ and vanish on $\partial\Omega$.

A comparison of (9.13) with the problem

$$(9.15) \qquad -\Delta z = \lambda_0'(\Omega) z \qquad \text{in } \Omega$$

$$z = 0 \qquad \text{on } \partial\Omega$$

$$z > 0 \qquad \text{in } \Omega$$

$$\Rightarrow \quad a_2 - \sigma_3 > \lambda_0'(\Omega) \Rightarrow \sigma_3 < a_2 - \lambda_0'(\Omega)$$

where $a_2$ is its local growth rate over $\Omega$ and $\lambda_0'(\Omega)$

is the principal eigenvalue of the negative Laplacian on $\Omega$ subject to Dirichlet (absorbing or lethal) homogeneous boundary conditions, reflecting a loss due to dissipation at the boundary of the habitat. Likewise

$$\sigma_4 < a_1 - \lambda_0^1(\Omega)$$

With only a bit more work, we can make a more substantial examination of the range of growth rates $a_1$ and $a_2$ for which (9.1) predicts competitive coexistence via permanence when the competition coefficients are held fixed. To this end, note first that the equilibrium density $\hat{w}_i(x)$ for species $i$ in the absence of species $j$ is completely determined by $a_i > \lambda_0^1(\Omega)$. So the inequalities

$$\lambda_+^1\left(a_2 - b_2 \hat{w}_1(x)\right) < 1$$

$$\lambda_+^1\left(a_1 - b_1 \hat{w}_2(x)\right) < 1$$

in (9.11) and (9.12) may be regarded for fixed values of

$b_1$ and $b_2$ as a coupled system of inequalities in $a_1$ and $a_2$, the solution of which gives the locus of permanence in those parameters.

Since
$$\begin{aligned} \Delta \tilde{w}_i + (a_i - \tilde{w}_i)\tilde{w}_i &= 0 && \text{in } \Omega \\ \tilde{w}_i &= 0 && \text{on } \partial\Omega \\ \tilde{w}_i &> 0 && \text{in } \Omega, \end{aligned}$$

$\sigma_1(a_i - \tilde{w}_i) = 0 \Rightarrow \lambda_+^1(a_i - \tilde{w}_i) = 1$. The monotonicity property of $\lambda_+^1(m)$ implies

$$\lambda_+^1(a_i - k\tilde{w}_i) < 1 \iff k < 1.$$

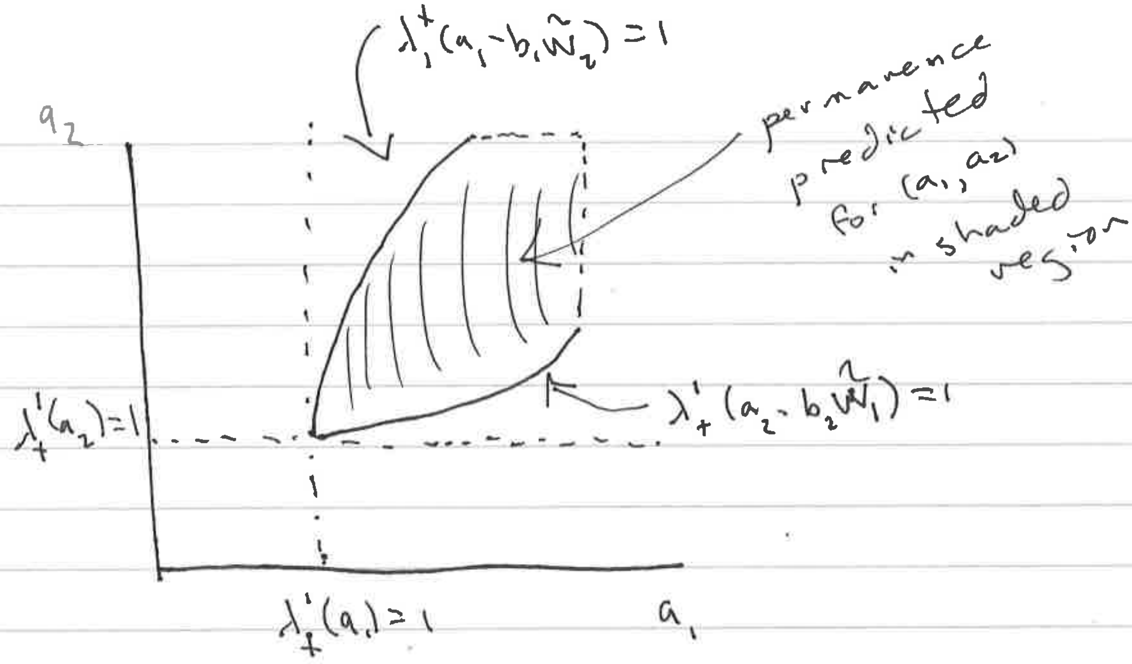
So if $a_1 = a_2 > \lambda_0^1(\Omega)$, we get permanence in (9.1) $\iff b_1 < 1$ and $b_2 < 1$.

So suppose now that $b_1 < 1$ and $b_2 < 1$ are fixed. The monotonicity property of $\lambda_+^1(m) \Rightarrow$ if $\lambda_+^1(a_i - b_i\tilde{w}_j) = 1$, then $\lambda_+^1(c - b_i\tilde{w}_j) < 1$ $\iff c > a_i$. Since $b_1 < 1$ and $b_2 < 1$,

we have

$\lambda_+^1(a_2 - b_1\tilde{w}_2) < 1$ and $\lambda_+^1(a_1 - b_2\tilde{w}_1) < 1$

So in order for $\lambda_+^1(a_1 - b_1\tilde{w}_2) = 1$, it must be the case that $a_1 < a_2$. In terms of an $a_1$-$a_2$ Cartesian plane the point $(a_1, a_2)$ for which $\lambda_+^1(a_1 - b_1\tilde{w}_2) = 1$ lies above the line $a_1 = a_2$. Similarly, for $\lambda_+^1(a_2 - b_2\tilde{w}_1) = 1$, it must be the case that $a_1 > a_2$, so that $(a_1, a_2)$ lies below the line $a_1 = a_2$. So the region $(a_1, a_2)$ of ordered pairs of local growth rates for which (9.1) is permanent for this fixed $b_1 < 1$ and $b_2 < 1$ is the open region between the two curves $\lambda_+^1(a_1 - b_1\tilde{w}_2) = 1$ and $\lambda_+^1(a_2 - b_2\tilde{w}_1) = 1$.




By working somewhat harder one may obtain estimates of the location of the boundary curves (Cantrell and Cosner 1987, Cantrell et al 1998)

References relevant to (9.1)

Cosner and Lazer (1984)
Cantrell and Cosner (1989)
Furter and Lopez-Gomez (1995, 1997)
Dancer (1991)
Eilbeck et al (1994)
Gui and Lou (1994)
Blat and Brown (1984)
Dancer (1984, 1985)
Korman and Leung (1986)
Leung (1980)
Pao (1981)
Schiafino and Tesei (1982)

# A Few Additional Comments on the Diffusive Logistic Model

Consider

$$\frac{\partial u}{\partial t} = \Delta u + (a-u)u \qquad \text{in } \Omega \times (0,\infty)$$

(9.16)

$$u = 0 \qquad \text{on } \partial\Omega \times (0,\infty)$$

we have observed that (9.16) has a globally attracting positive equilibrium $u_a^*$ for $a > \lambda_0^1(\Omega)$ and that $u_a^* \leq a$ on $\overline{\Omega}$.

Notice that if $a < a'$,

$$\Delta u_a^* + (a' - u_a^*)u_a^*$$

$$= \Delta u_a^* + (a - u_a^*)u_a^* + (a' - a)u_a^*$$

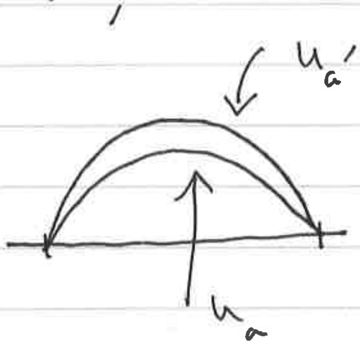
$$= (a' - a)u_a^* > 0 \quad \text{in } \Omega \qquad (= 0 \text{ on } \partial\Omega)$$

It follows that $u_a$ is a lower solution for $\Delta u + (a' - u)u^* = 0$

Consequently, since any constant $K > a'$ is an upper solution, we have that

$$u_a^* \leq u_{a'}^*$$

and indeed $u_a^* < u_{a'}^*$ in $\Omega$. So $u_a$ is strictly increasing in $a$ ( in the sense that is possible given Dirichlet boundary conditions, $u_a^*(x) < u_{a'}^*(x)$ in $\Omega$




with $\frac{\partial u_{a'}^*}{\partial \eta} < \frac{\partial u_a^*}{\partial \eta}$ on $\partial\Omega$ )

Suppose now $u_{a_n}^* \to 0$ on $\bar{\Omega}$ uniformly where $a_n \to a^*$. Now

$$-\Delta u_{a_n}^* = (a_n - u_{a_n}^*) u_{a_n}^* \quad \text{in } \Omega$$

Divide $u_{a_n}^*$ by its supremum norm, say, and let $w_{a_n} = u_{a_n}^* / \|u_{a_n}^*\|_\infty$

Then

$$-\Delta w_{a_n} = w_{a_n}(a_n - u_{a_n^*}) \quad \text{in } \Omega$$
(9.17)
$$w_{a_n} = 0 \quad \text{on } \partial\Omega$$

It follows from continuity of $(-\Delta)^{-1}$ and embedding theorems that $w_{a_n}$ in (9.17) and the original $u^*_{a_n}$ in (9.16) are bounded in, say, $C^{1,\alpha}_0(\bar\Omega)$ and thus both may be assumed convergent in $C^\alpha_0(\bar\Omega)$ (passing to a subsequence if need be).

Clearly $\lim_{a_n \to a^*} u_{a_n^*} = 0$. Let $\lim_{a_n \to a^*} w_{a_n} = w$.

It follows from the continuity of $(-\Delta)^{-1}$, that $w_{a_n} \to w$ in $C^{2+\alpha}_0(\bar\Omega)$ and that

$$-\Delta w = a^* w \quad \text{in } \Omega$$
$$w = 0 \quad \text{on } \partial\Omega$$

Clearly $w_{a_n} \Rightarrow w$ in $C^{2+\alpha}(\bar\Omega) \Longrightarrow$

$w_{a_n} \to w$ in $C(\Omega)$ as well. So $w \geq 0$ and $\|w\|_\infty = 1$. So $w \not\equiv 0$. So $w > 0$ in $\Omega$. It follows that $a^* = \lambda_0^1(\Omega)$